

On local compactness in quasilinear elliptic problems

K. Adriouch and A. El Hamidi

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Abstract

One of the major difficulties in nonlinear elliptic problems involving critical nonlinearities is the compactness of Palais-Smale sequences. In their celebrated work [7], Brézis and Nirenberg introduced the notion of critical level for these sequences in the case of a critical perturbation of the Laplacian homogeneous eigenvalue problem. In this paper, we give a natural and general formula of the critical level for a large class of nonlinear elliptic critical problems. The sharpness of our formula is established by the construction of suitable Palais-Smale sequences which are not relatively compact.

1 Introduction

In nonlinear elliptic variational problems involving critical nonlinearities, one of the major difficulties is to recover the compactness of Palais-Smale sequences of the associated Euler-Lagrange functional. Such questions were first studied, in our knowledge, by Brézis and Nirenberg in their well-known work [7]. The concentration-compactness principle due to Lions [12] is widely used to overcome these difficulties. Other methods, based on the convergence almost everywhere of the gradients of Palais-Smale sequences, can be also used to recover the compactness. We refer the reader to the papers by Boccardo and Murat [5] and by J. M. Rakotoson [14] for bounded domains.

For arbitrary domains, we refer to the recent work by A. El Hamidi and J. M. Rakotoson [9].

In [7], the authors studied the critical perturbation of the eigenvalue problem:

$$\begin{cases} -\Delta u = \lambda u + u^{2^*-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where Ω is a bounded domain in \mathbb{R}^N , $N \geq 3$, with smooth boundary, $2^* = \frac{2N}{N-2}$ is the Sobolev critical exponent of the embedding $W^{1,2}(\Omega) \subset L^p(\Omega)$, and λ is a positive parameter. The authors introduced an important condition on the level corresponding to the energy of Palais-Smale sequences which guarantees their relative compactness. Indeed, let (u_n) be a Palais-Smale sequence for the Euler-Lagrange functional

$$I_\lambda(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \frac{\lambda}{2} \int_{\Omega} |u|^2 - \frac{1}{2^*} \int_{\Omega} |u|^{2^*}.$$

More precisely, the authors showed that if

$$\lim_{n \rightarrow +\infty} I_\lambda(u_n) < \frac{1}{N} S^{\frac{N}{2}} \quad (1.2)$$

then (u_n) is relatively compact, which implies the existence of nontrivial critical points of I_λ . Here, S denotes the best Sobolev constant in the embedding $W_0^{1,2}(\Omega) \subset L^{2^*}(\Omega)$. In this work, we begin by giving the generalization of condition (1.2) for the quasilinear equation

$$\begin{aligned} -\Delta_p u &= \lambda f(x, u) + |u|^{p^*-2} u & \text{in } \Omega, \\ u|_{\Gamma} &= 0 \quad \text{and} \quad \frac{\partial u}{\partial \nu}|_{\Sigma} = 0, \end{aligned} \quad (1.3)$$

where Ω is a bounded domain in \mathbb{R}^N , $N \geq 3$, with smooth boundary $\partial\Omega = \overline{\Gamma} \cup \overline{\Sigma}$, where Γ and Σ are smooth $(N-1)$ -dimensional submanifolds of $\partial\Omega$ with positive measures such that $\Gamma \cap \Sigma = \emptyset$. Δ_p is the p -Laplacian and $\frac{\partial}{\partial \nu}$ is the outer normal derivative. Here, f is a subcritical perturbation of $|u|^{p^*-1}$.

The sharpness of our result is established by the construction of suitable Palais-Smale sequences (corresponding to the critical level) which are not relatively compact.

Then we give the analogous condition to (1.2) for a general system with critical exponents

$$\begin{cases} -\Delta_p u = \lambda f(x, u) + u|u|^{\alpha-1}|v|^{\beta+1} & \text{in } \Omega \\ -\Delta_q v = \mu g(x, v) + |u|^{\alpha+1}|v|^{\beta-1}v & \text{in } \Omega \end{cases}$$

together with Dirichlet or mixed boundary conditions, where f and g are subcritical perturbations of $|u|^{p^*-1}$ and $|v|^{q^*-1}$ respectively, $p^* = \frac{Np}{N-p}$ (resp. $q^* = \frac{Nq}{N-q}$) is the critical exponent of the Sobolev embedding $W^{1,p}(\Omega) \subset L^r(\Omega)$ (resp. $W^{1,q}(\Omega) \subset L^r(\Omega)$). Our approach provides a general condition based on the Nehari manifold, which can be extended to a large class of critical nonlinear problems. In this work, we confine ourselves to systems involving (p, q) -Laplacian operators and critical nonlinearities. The sharpness of our result is established, in the special case $p = q$, by the construction of suitable Palais-Smale sequences which are not relatively compact. The question of sharpness corresponding to the case $p \neq q$ is still open.

For a more complete description of nonlinear elliptic systems, we refer the reader to the papers by De Figueiredo [10] and by De Figueiredo & Felmer [11] and the references therein.

2 A general local compactness result

For the reader's convenience, we start with the scalar case and to render the paper selfcontained we will recall or show some well-known facts.

2.1 The scalar case

Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be a bounded domain with smooth boundary $\partial\Omega$. Let $f(x, u) : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a function which is measurable in x , continuous in u and satisfying the growth condition at infinity

$$|f(x, u)| = o(u^{p^*-1}) \text{ as } u \rightarrow +\infty, \text{ uniformly in } x. \quad (2.4)$$

This situation occurs, for example, in the special cases $f(x, u) = u$ or $f(x, u) = u^{q-1}$, $1 < q < p^*$.

Consider the problem

$$\begin{aligned} -\Delta_p u &= \lambda f(x, u) + |u|^{p^*-2}u \text{ in } \Omega, \\ u|_{\Gamma} &= 0 \text{ and } \frac{\partial u}{\partial \nu}|_{\Sigma} = 0, \end{aligned} \quad (2.5)$$

where Ω is a bounded domain in \mathbb{R}^N , $N \geq 3$, with smooth boundary $\partial\Omega = \overline{\Gamma} \cup \overline{\Sigma}$, where Γ and Σ are smooth $(N-1)$ -dimensional submanifolds of $\partial\Omega$ with positive measures such that $\Gamma \cap \Sigma = \emptyset$. Problem (2.5) is posed in the framework of the Sobolev space

$$W_{\Gamma}^{1,p}(\Omega) = \{u \in W^{1,p}(\Omega) : u|_{\Gamma} = 0\},$$

which is the closure of $C_0^1(\Omega \cap \Gamma, \mathbb{R})$ with respect to the norm of $W^{1,p}(\Omega)$. Notice that $\text{meas}(\Gamma) > 0$ implies that the Poincaré inequality is still available in $W_{\Gamma}^{1,p}(\Omega)$, so it can be endowed with the norm

$$\|u\| = \|\nabla u\|_p$$

and $(W_{\Gamma}^{1,p}(\Omega), \|\cdot\|)$ is a reflexive and separable Banach space. The associated Euler-Lagrange functional is given by

$$J_{\lambda}(u) := \frac{1}{p} \|\nabla u\|_p^p - \frac{1}{p^*} \|u\|_{p^*}^{p^*} - \lambda \int_{\Omega} F(x, u(x)) dx$$

the corresponding Euler-Lagrange functional, where $F(x, u) := \int_0^u f(x, s) ds$.

We recall here that the Nahari manifold associated to the functional J_λ is given by:

$$\mathcal{N}_{J_\lambda} = \{u \in W_\Gamma^{1,p}(\Omega) \setminus \{0\} : J'_\lambda(u)(u) = 0\},$$

and it is clear that \mathcal{N}_{J_λ} contains all nontrivial critical points of J_λ . This manifold can be characterized more explicitly by the following

$$\mathcal{N}_{J_\lambda} = \left\{ tu, (t, u) \in (\mathbb{R} \setminus \{0\}) \times (W_\Gamma^{1,p}(\Omega) \setminus \{0\}) : \frac{d}{dt} J_\lambda(tu) = 0 \right\},$$

where $t \mapsto J_\lambda(tu)$ is a function defined from \mathbb{R} to itself, for every u given in $W_\Gamma^{1,p}(\Omega) \setminus \{0\}$. We define the critical level associated to Problem (2.5) by:

$$c^*(\lambda) := \inf_{w \in \mathcal{N}_{J_0}} J_0(w) + \inf_{w \in \mathcal{N}_{J_\lambda} \cup \{0\}} J_\lambda(w). \quad (2.6)$$

At this stage, we can state and show our first result

Theorem 2.1 *Let $\lambda \in \mathbb{R}$ and (u_n) be a Palais-Smale sequence of J_λ such that*

$$\lim_{n \rightarrow +\infty} J_\lambda(u_n) < c^*(\lambda). \quad (2.7)$$

Then (u_n) is relatively compact.

Proof. Let $\lambda \in \mathbb{R}$ and (u_n) be a Palais-Smale sequence for J_λ of level $c \in \mathbb{R}$ ((PS) $_c$ for short) satisfying the condition (2.7). We claim that (u_n) is bounded in $W_\Gamma^{1,p}(\Omega)$. Indeed, on has one hand

$$\frac{1}{p} \|\nabla u_n\|_p^p - \frac{1}{p^*} \|u_n\|_{p^*}^{p^*} - \lambda \int_\Omega F(x, u_n) dx = c + o_n(1), \quad (2.8)$$

and

$$\|\nabla u_n\|_p^p - \|u_n\|_{p^*}^{p^*} - \lambda \int_\Omega f(x, u_n) u_n dx = o_n(\|\nabla u_n\|_p). \quad (2.9)$$

Then,

$$\left(\frac{1}{p} - \frac{1}{p^*} \right) \|u_n\|_{p^*}^{p^*} + \frac{\lambda}{p} \int_\Omega f(x, u_n) u_n dx - \lambda \int_\Omega F(x, u_n) dx = c + o_n(1) + o_n(\|\nabla u_n\|_p).$$

Now, let $\varepsilon > 0$, using the growth condition (2.4), there exists $c_1(\varepsilon) > 0$ such that

$$|f(x, u)| \leq \varepsilon |u|^{p^*-1} + c_1 \text{ and } |F(x, u)| \leq \frac{\varepsilon}{p^*} |u|^{p^*} + c_1, \text{ a.e. } x \in \Omega \text{ and for every } u \in \mathbb{R}.$$

Applying the Hölder and the Young inequalities to the last relations, it follows

$$\|u_n\|_{p^*}^{p^*} \leq \varepsilon \|\nabla u_n\|_p + c_2(|\Omega|, \lambda, \varepsilon). \quad (2.10)$$

Combining (2.10) and (2.8), we deduce that (u_n) is in fact bounded in $W_\Gamma^{1,p}(\Omega)$. So passing, if necessary to a subsequence, we can consider that

$$\begin{aligned} u_n &\rightharpoonup u \text{ in } W_\Gamma^{1,p}(\Omega), \\ u_n &\rightarrow u \text{ a.e. in } \Omega. \end{aligned}$$

On the other hand, the growth condition (2.4) implies also that, for almost every $x \in \Omega$, the functions $s \mapsto F(x, s)$ and $s \mapsto sf(x, s)$ satisfy the conditions of the Brézis-Lieb Lemma (see Theorem 2 in [6]). Thus, we get the identities

$$\begin{aligned} \int_\Omega F(x, v_n) dx &= \int_\Omega F(x, u_n) - \int_\Omega F(x, u) + o_n(1), \\ \int_\Omega f(x, v_n) v_n dx &= \int_\Omega f(x, u_n) u_n - \int_\Omega f(x, u) u + o_n(1). \end{aligned}$$

Moreover, let $\varepsilon > 0$, there is $c_1(\varepsilon) > 0$ such that

$$\left| \int_\Omega f(x, v_n) v_n dx \right| \leq \varepsilon \|v_n\|_{p^*}^{p^*} + c_1 \|v_n\|_1.$$

Let $C > 0$ (which is independent of n and ε), such that $\|v_n\|_{p^*}^{p^*} \leq C$. Since (v_n) converges strongly to 0 in $L^1(\Omega)$, there is $n_0(\varepsilon) \in \mathbb{N}$ such that $\|v_n\|_1 \leq \varepsilon/c_1$, for every $n \geq n_0(\varepsilon)$, and consequently

$$\left| \int_\Omega f(x, v_n) v_n dx \right| \leq \varepsilon(1 + C), \quad \forall n \geq n_0(\varepsilon).$$

In the same way, rewriting $F(x, v_n) = \int_0^{v_n} f(x, s) ds$ and using the same arguments as above, we deduce that

$$\int_{\Omega} F(x, v_n) dx = o_n(1) \quad (2.11)$$

$$\int_{\Omega} f(x, v_n) v_n dx = o_n(1). \quad (2.12)$$

Applying once again the Brézis-Lieb Lemma, we conclude that $u \in \mathcal{N}_{J_{\lambda}} \cup \{0\}$ and

$$\|v_n\|^p - \|v_n\|_{p^*}^{p^*} = o_n(1), \quad (2.13)$$

$$J_0(v_n) := \frac{1}{p} \|v_n\|^p - \frac{1}{p^*} \|v_n\|_{p^*}^{p^*} = c - J_{\lambda}(u) + o_n(1). \quad (2.14)$$

A direct computation gives

$$\mathcal{N}_{J_0} = \{t_0(u)u : u \in W_{\Gamma}^{1,p}(\Omega) \setminus \{0\}\},$$

where

$$t_0(u) := \left(\frac{\|u\|^p}{\|u\|_{p^*}^{p^*}} \right)^{\frac{1}{p^*-p}}.$$

Now, let b be the common limit of $\|v_n\|^p$ and $\|v_n\|_{p^*}^{p^*}$. Suppose that $b \neq 0$.

On one hand we have

$$\begin{aligned} J_0(t_0(v_n)v_n) &= \left(\frac{1}{p} - \frac{1}{p^*} \right) \left(\frac{\|v_n\|^p}{\|v_n\|_{p^*}^{p^*}} \right)^{\frac{p^*}{p^*-p}} \\ &\geq \inf_{w \in \mathcal{N}_{J_0}} J_0(w). \end{aligned}$$

Then

$$\lim_{n \rightarrow +\infty} J_0(t_0(v_n)v_n) = \frac{b}{N} \geq \inf_{w \in \mathcal{N}_{J_0}} J_0(w).$$

On the other hand, the identity (2.14) leads to

$$\frac{b}{N} = c - J_{\lambda}(u).$$

It follows then

$$\begin{aligned} c &\geq \inf_{w \in \mathcal{N}_{J_0}} J_0(w) + J_\lambda(u) \\ &\geq \inf_{w \in \mathcal{N}_{J_0}} J_0(w) + \inf_{w \in \mathcal{N}_{J_\lambda} \cup \{0\}} J_\lambda(w), \end{aligned}$$

which contradicts the condition (2.7). This achieves the proof. \square

2.2 Sharpness of the critical level formula in the scalar case

To show the sharpness of the critical level formula (2.7), it suffices to carry out a Palais-Smale sequence for J_λ of level $c^*(\lambda)$ which contains no convergent subsequence.

Consider, for a given $\varepsilon > 0$, the extremal function

$$\Phi_\varepsilon(x) = C_N \varepsilon^{\frac{N-p}{p^2}} \left(\varepsilon + |x|^{\frac{p}{p-1}} \right)^{\frac{p-N}{p}} \quad \text{with} \quad C_N := \left(N \left(\frac{N-p}{p-1} \right)^{p-1} \right)^{(N-p)/p^2}$$

which attains the best constant S of the Sobolev embedding

$$D^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N).$$

Without loss of generality, we can consider that $0 \in \Sigma$. Moreover, the set $\partial\Omega$ satisfies the following property (see more details in Adimurthi, Pacella and Yadava [1]):

There exist $\delta > 0$, an open neighborhood \mathcal{V} of 0 and a diffeomorphism $\Psi : B_\delta(0) \rightarrow \mathcal{V}$ which has a jacobian determinant equal to one at 0, with $\Psi(B_\delta^+) = \mathcal{V} \cap \Omega$, where $B_\delta^+ = B_\delta(0) \cap \{x \in \mathbb{R}^N : x_N > 0\}$.

Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $\varphi \equiv 1$ in a neighborhood of the origin.

We define the sequence defined by

$$\psi_n(x) := \varphi(x) \Phi_{1/n}(x), \quad \text{for } n \in \mathbb{N}^*. \quad (2.15)$$

It is well known that the sequence $(\psi_n) \subset W_\Gamma^{1,p}(\Omega)$ is a Palais-Smale sequence for J_0 of level $\inf_{w \in \mathcal{N}_{J_0}} J_0(w)$, which satisfies

$$\begin{aligned} \psi_n &\rightarrow 0 \text{ a.e. in } \Omega, \\ \nabla \psi_n &\rightarrow 0 \text{ a.e. in } \Omega, \\ \|\psi_n\|_{p^*}^{p^*} &\longrightarrow \left[N \inf_{w \in \mathcal{N}_{J_0}} J_0(w) \right]^{p/N} := \ell \text{ as } n \longrightarrow +\infty, \\ \|\nabla \psi_n\|_p^p &\longrightarrow \left[N \inf_{w \in \mathcal{N}_{J_0}} J_0(w) \right]^{p/N} := \ell \text{ as } n \longrightarrow +\infty. \end{aligned}$$

Now, let (u_n) be a Palais-Smale sequence of J_λ of level $\inf_{w \in \mathcal{N}_{J_\lambda} \cup \{0\}} J_\lambda(w)$. We will not go into further details concerning which subcritical terms $f(u)$ allow the existence of such sequences, but in the litterature, this occurs for various classes of subcritical terms. Applying Theorem 2.1, there exists a subsequence, still denoted by (u_n) , which converges to some $u \in W_\Gamma^{1,p}(\Omega)$. Then

$$\begin{aligned} \|u_n + \psi_n\|_{p^*} &\leq C, \\ u_n + \psi_n &\rightarrow u \text{ a.e. in } \Omega, \\ \|\nabla u_n + \nabla \psi_n\|_p &\leq C, \\ \nabla u_n + \nabla \psi_n &\rightarrow \nabla u \text{ a.e. in } \Omega. \end{aligned}$$

where C a positive constant independent of n . We apply the Brézis-Lieb Lemma to the sequence $(u_n + \psi_n)$ and get

$$\|u_n + \psi_n\|_{p^*}^{p^*} = \|(u_n - u) + \psi_n\|_{p^*}^{p^*} + \|u\|_{p^*}^{p^*} + o_n(1).$$

Moreover, one has

$$-\|u_n - u\|_{p^*} + \|\psi_n\|_{p^*} - \ell^{1/p^*} \leq \|(u_n - u) + \psi_n\|_{p^*} - \ell^{1/p^*} \leq \|u_n - u\|_{p^*} + \|\psi_n\|_{p^*} - \ell^{1/p^*}$$

which implies that

$$\|(u_n - u) + \psi_n\|_{p^*} - \ell^{1/p^*} = o_n(1).$$

Therefore, we conclude that

$$\|u_n + \psi_n\|_{p^*}^{p^*} = \|u\|_{p^*}^{p^*} + \ell + o_n(1).$$

The same arguments applied to the sequence $(\nabla u_n + \nabla \psi_n)$ give

$$\|\nabla u_n + \nabla \psi_n\|_p^p = \|\nabla u\|_p^p + \ell + o_n(1).$$

Finally, using the fact that

$$|\psi_n|^{p^*} \rightharpoonup \ell \delta_0 \text{ weakly* in } \mathcal{M}^+(\Omega) \quad (2.16)$$

$$|\nabla \psi_n|^p \rightharpoonup \ell \delta_0 \text{ weakly* in } \mathcal{M}^+(\Omega) \quad (2.17)$$

where δ_0 is the Dirac measure concentrated at the origin and $\mathcal{M}^+(\Omega)$ is the space of positive finite measures [20]), we get that the sequence $(u_n + \psi_n)$ is a Palais-Smale sequence of J_λ of level $c^*(\lambda)$.

We hence constructed a Palais-Smale sequence $(u_n + \psi_n)$ of J_λ of level $c^*(\lambda)$ which can not be relatively compact in $W_\Gamma^{1,p}(\Omega)$. This justifies the sharpness of the critical level formula (2.7).

Remark 2.1 *If we are interested by the homogeneous Dirichlet conditions, i.e. if $\Sigma = \emptyset$, the same arguments developed above are still valid, it suffices to assume that the origin $0 \in \Omega$ and consider $\varphi \in C_0^\infty(\Omega)$ such that $\varphi \equiv 1$ in a neighborhood of the origin.*

2.3 The system case

Now, consider the system

$$\begin{cases} -\Delta_p u = \lambda f(x, u) + u|u|^{\alpha-1}|v|^{\beta+1}, \\ -\Delta_q v = \mu g(x, v) + |u|^{\alpha+1}|v|^{\beta-1}v, \end{cases} \quad (2.18)$$

together with Dirichlet or mixed boundary conditions

$$\begin{cases} u|_{\Gamma_1} = 0 \text{ and } \frac{\partial u}{\partial \nu}|_{\Sigma_1} = 0, \\ v|_{\Gamma_2} = 0 \text{ and } \frac{\partial v}{\partial \nu}|_{\Sigma_2} = 0, \end{cases} \quad (2.19)$$

where, Ω is a bounded domain in \mathbb{R}^N , $N \geq 3$, with smooth boundary $\partial\Omega = \overline{\Gamma}_1 \cup \overline{\Sigma}_i$, where Γ_i and Σ_i are smooth $(N-1)$ -dimensional submanifolds of $\partial\Omega$ with positive measures such that $\Gamma_i \cap \Sigma_i = \emptyset$, $i \in \{1, 2\}$. Δ_p is the p -Laplacian and $\frac{\partial}{\partial \nu}$ is the outer normal derivative. Also, it is clear that when $\Gamma_1 = \Gamma_2 = \partial\Omega$, one deals with homogeneous Dirichlet boundary conditions. We assume here that

$$1 < p < N, \quad 1 < q < N, \quad (2.20)$$

and the critical condition

$$\frac{\alpha+1}{p^*} + \frac{\beta+1}{q^*} = 1. \quad (2.21)$$

Indeed, this condition represents the maximal growth such that the integrability of the product term $|u|^{\alpha+1}|v|^{\beta+1}$ (which will appear in the Euler-Lagrange functional) can be guaranteed by suitable Hölder estimates.

The functions f and g are two caratheodory functions which satisfy the growth conditions

$$|f(x, u)| = o(u^{p^*-1}) \text{ as } u \rightarrow +\infty, \text{ uniformly in } x, \quad (2.22)$$

$$|g(x, v)| = o(v^{q^*-1}) \text{ as } v \rightarrow +\infty, \text{ uniformly in } x. \quad (2.23)$$

Problem (2.18), together with (2.19), is posed in the framework of the Sobolev space $W = W_{\Gamma_1}^{1,p}(\Omega) \times W_{\Gamma_2}^{1,q}(\Omega)$, where

$$W_{\Gamma_1}^{1,p}(\Omega) = \{u \in W^{1,p}(\Omega) : u|_{\Gamma_1} = 0\}, \quad W_{\Gamma_2}^{1,q}(\Omega) = \{u \in W^{1,q}(\Omega) : u|_{\Gamma_2} = 0\},$$

which are respectively the closure of $C_0^1(\Omega \cap \Gamma_1, \mathbb{R})$ with respect to the norm of $W^{1,p}(\Omega)$ and $C_0^1(\Omega \cap \Gamma_2, \mathbb{R})$ with respect to the norm of $W^{1,q}(\Omega)$. Notice that $\text{meas}(\Gamma_i) > 0$, $i = 1, 2$, imply that the Poincaré inequality is still available in $W_{\Gamma_1}^{1,p}(\Omega)$ and $W_{\Gamma_2}^{1,q}(\Omega)$, so W can be endowed with the norm

$$\|(u, v)\| = \|\nabla u\|_p + \|\nabla v\|_q$$

and $(W, \|\cdot\|)$ is a reflexive and separable Banach space. The associated Euler-Lagrange functional $I_{\lambda,\mu} \in C^1(W, \mathbb{R})$ is given by

$$I_{\lambda,\mu}(u, v) = (\alpha+1) \left(\frac{P(u)}{p} - \lambda \int_{\Omega} F(x, u) \right) + (\beta+1) \left(\frac{Q(v)}{q} - \mu \int_{\Omega} G(x, v) \right) - R(u, v),$$

where $P(u) = \|\nabla u\|_p^p$, $Q(v) = \|\nabla v\|_q^q$, $F(x, u) = \int_0^u f(x, s) ds$, $G(x, v) = \int_0^v g(x, t) dt$, and $R(u, v) = \int_{\Omega} |u|^{\alpha+1} |v|^{\beta+1} dx$. Notice that $R(u, v) \leq \|u\|_{p^*}^{\alpha+1} \|v\|_{q^*}^{\beta+1} < +\infty$.

Consider the Nehari manifold associated to Problem (2.18) given by

$$\mathcal{N}_{\lambda,\mu} = \{(u, v) \in W \setminus \{(0, 0)\} / D_1 I_{\lambda,\mu}(u, v)(u) = D_2 I_{\lambda,\mu}(u, v)(v) = 0\},$$

where $D_1 I_{\lambda,\mu}$ and $D_2 I_{\lambda,\mu}$ are the derivative of $I_{\lambda,\mu}$ with respect to the first variable and the second variable respectively.

An interesting and useful characterization of $\mathcal{N}_{\lambda,\mu}$ is the following

$$\mathcal{N}_{\lambda,\mu} = \{(su, tv) / (s, u, t, v) \in \mathcal{Z}^* \text{ and } \partial_s I_{\lambda,\mu}(su, tv) = \partial_t I_{\lambda,\mu}(su, tv) = 0\},$$

where

$$\mathcal{Z}^* = \{(s, u, t, v); (s, t) \in \mathbb{R}^2, (u, v) \in W_{\Gamma_1}^{1,p}(\Omega) \times W_{\Gamma_2}^{1,q}(\Omega), (su, tv) \neq (0, 0)\}$$

and $I_{\lambda,\mu}$ is considered as a functional of four variables (s, u, t, v) in $\mathcal{Z} := \mathbb{R} \times W_{\Gamma_1}^{1,p}(\Omega) \times \mathbb{R} \times W_{\Gamma_2}^{1,q}(\Omega)$.

Definition 2.1 Let λ and μ be two real parameters. A sequence $(u_n, v_n) \in W$ is a Palais-Smale sequence of the functional $I_{\lambda,\mu}$ if

- there exists $c \in \mathbb{R}$ such that $\lim_{n \rightarrow +\infty} I_{\lambda,\mu}(u_n, v_n) = c$ (2.24)

- $DI_{\lambda,\mu}(u_n, v_n)$ converges strongly in the dual W' of W (2.25)

where $DI_{\lambda,\mu}(u_n, v_n)$ denotes the Gâteaux derivative of $I_{\lambda,\mu}$.

The last condition (2.25) implies that

$$D_1 I_{\lambda,\mu}(u_n, v_n)(u_n) = o(\|u_n\|_{p^*}) \quad (2.26)$$

$$D_2 I_{\lambda,\mu}(u_n, v_n)(v_n) = o(\|v_n\|_{q^*}). \quad (2.27)$$

where $D_1 I_{\lambda,\mu}(u_n, v_n)$ (resp. $D_2 I_{\lambda,\mu}(u_n, v_n)$) denotes the Gâteaux derivative of $I_{\lambda,\mu}$ with respect to its first (resp. second) variable.

We introduce the critical level corresponding to Problem (2.18) by

$$c^*(\lambda, \mu) := \inf_{w \in \mathcal{N}_{0,0}} I_{0,0}(w) + \inf_{w \in \mathcal{N}_{\lambda,\mu} \cup \{(0,0)\}} I_{\lambda,\mu}(w). \quad (2.28)$$

Then we have the following

Theorem 2.2 *Let λ and μ be two real parameters and (u_n, v_n) be a Palais-Smale sequence of $I_{\lambda,\mu}$ such that*

$$c := \lim_{n \rightarrow +\infty} I_{\lambda,\mu}(u_n, v_n) < c^*(\lambda, \mu). \quad (2.29)$$

Then (u_n, v_n) relatively compact.

Proof. Let λ and μ be two real parameters and (u_n, v_n) be a Palais-Smale sequence of $I_{\lambda,\mu}$ satisfying the condition (2.29). We claim that (u_n, v_n) is bounded in W . Indeed, on one hand conditions (2.24), (2.26) and (2.27) can be rewritten as the following

$$I_{\lambda,\mu}(u_n, v_n) = c + o_n(1) \quad (2.30)$$

$$P(u_n) - \lambda \int_{\Omega} f(x, u_n) u_n dx = R(u_n, v_n) + o(\|u_n\|_{p^*}) \quad (2.31)$$

$$Q(v_n) - \mu \int_{\Omega} f(x, v_n) v_n dx = R(u_n, v_n) + o(\|v_n\|_{q^*}). \quad (2.32)$$

Using (2.21), one gets

$$\begin{aligned} R(u_n, v_n) &= \frac{\alpha+1}{p^*} \left(P(u_n) - \lambda \int_{\Omega} f(x, u_n) u_n \right) + o(\|u_n\|_{p^*}) \\ &+ \frac{\beta+1}{q^*} \left(Q(v_n) - \mu \int_{\Omega} g(x, v_n) v_n \right) + o(\|v_n\|_{q^*}). \end{aligned} \quad (2.33)$$

Suppose that there is a subsequence, still denoted by (u_n, v_n) in W which is unbounded, *i.e.* $\|\nabla u_n\|_p + \|\nabla v_n\|_q$ tends to $+\infty$ as n goes to $+\infty$.

If

$$\lim_{n \rightarrow +\infty} \|\nabla u_n\|_p = +\infty,$$

then using (2.22) one has

$$\begin{aligned} \int_{\Omega} |f(x, u_n)u_n| &= o(P(u_n)), \\ \int_{\Omega} |F(x, u_n)| &= o(P(u_n)), \end{aligned}$$

since (2.22) implies that for every $\varepsilon > 0$, there exists $c_1(\varepsilon) > 0$ such that

$$|f(x, s)| \leq \varepsilon |s|^{p^*-1} + c_1 \text{ and } |F(x, s)| \leq \frac{\varepsilon}{p^*} |s|^{p^*} + c_1, \text{ a.e. } x \in \Omega, \forall s \in \mathbb{R}.$$

Similarly, if

$$\lim_{n \rightarrow +\infty} \|\nabla v_n\|_q = +\infty,$$

then using (2.23) it follows

$$\begin{aligned} \int_{\Omega} |g(x, v_n)v_n| &= o(Q(v_n)), \\ \int_{\Omega} |G(x, v_n)| &= o(Q(v_n)). \end{aligned}$$

On one hand, suppose that

$$\lim_{n \rightarrow +\infty} \|\nabla u_n\|_p = \lim_{n \rightarrow +\infty} \|\nabla v_n\|_q = +\infty.$$

Substituting (2.33) in (2.30), we obtain

$$\begin{aligned} c + o_n(1) &= (\alpha + 1) \left(\frac{1}{p} - \frac{1}{p^*} + o(P(u_n))^{\frac{p^*-p}{p}} \right) P(u_n) \\ &+ (\beta + 1) \left(\frac{1}{q} - \frac{1}{q^*} + o(Q(v_n))^{\frac{q^*-q}{q}} \right) Q(v_n) \xrightarrow{n \rightarrow +\infty} +\infty \end{aligned}$$

which can not hold true. On the other hand, suppose that

$$\lim_{n \rightarrow +\infty} \|\nabla u_n\|_p = +\infty \text{ and the sequence } \|\nabla v_n\|_q \text{ is bounded,}$$

then (2.31) implies that $R(u_n, v_n)$ is unbounded while (2.32) implies, on the contrary, that $R(u_n, v_n)$ is bounded. The case

$$\lim_{n \rightarrow +\infty} \|\nabla v_n\|_q = +\infty \text{ and the sequence } \|\nabla u_n\|_p \text{ is bounded,}$$

leads to a contradiction with the same argument, which achieves the claim.

At this stage, we can assume, up to a subsequence, that

$$\begin{aligned} u_n &\rightharpoonup u \text{ in } W_{\Gamma_1}^{1,p}(\Omega), \\ v_n &\rightharpoonup v \text{ in } W_{\Gamma_2}^{1,q}(\Omega), \\ u_n &\rightarrow u \text{ a.e. in } \Omega, \\ v_n &\rightarrow v \text{ a.e. in } \Omega. \end{aligned}$$

It is clear that

$$(u, v) \in \mathcal{N}_{\lambda, \mu} \cup \{(0, 0)\}.$$

Let us set

$$X_n = u_n - u \text{ and } Y_n = v_n - v.$$

Using again the growth conditions (2.22) and (2.23), we show easily that the functions, which are defined on $\Omega \times \mathbb{R}$: $(x, s) \mapsto sf(x, s)$, $(x, s) \mapsto sg(x, s)$, $(x, s) \mapsto F(x, s)$ and $(x, s) \mapsto G(x, s)$ satisfy the conditions of the Brézis-Lieb lemma [6]. Then, we have the decompositions

$$\begin{aligned} \int_{\Omega} F(x, X_n) &= \int_{\Omega} F(x, u_n) - \int_{\Omega} F(x, u) + o_n(1), \\ \int_{\Omega} f(x, X_n) X_n &= \int_{\Omega} f(x, u_n) u_n - \int_{\Omega} f(x, u) u + o_n(1), \\ \int_{\Omega} G(x, Y_n) &= \int_{\Omega} G(x, v_n) - \int_{\Omega} G(x, v) + o_n(1), \\ \int_{\Omega} g(x, Y_n) Y_n &= \int_{\Omega} g(x, v_n) v_n - \int_{\Omega} g(x, v) v + o_n(1). \end{aligned}$$

Moreover, let $\varepsilon > 0$, then there is $c_1(\varepsilon) > 0$ such that

$$\left| \int_{\Omega} f(x, X_n) X_n dx \right| \leq \varepsilon \|X_n\|_{p^*}^{p^*} + c_1 \|X_n\|_1.$$

Let C be a positive constant such that $\|X_n\|_{p^*}^{p^*} \leq C$. Since X_n converges to 0 in $L^1(\Omega)$, there exists $n_0(\varepsilon) \in \mathbb{N}$ verifying $\|X_n\|_1 \leq \varepsilon/c_1$, for every $n \geq n_0(\varepsilon)$, thus

$$\left| \int_{\Omega} f(x, X_n) X_n \, dx \right| \leq \varepsilon(1 + C), \quad \forall n \geq n_0(\varepsilon).$$

In the same manner, writing $F(x, X_n) = \int_0^{X_n} f(x, s) \, ds$ and using the same arguments as above, we get

$$\int_{\Omega} F(x, X_n) = o_n(1) \quad \text{and} \quad \int_{\Omega} f(x, X_n) X_n = o_n(1).$$

Similarly, it follows that

$$\int_{\Omega} G(x, Y_n) = o_n(1) \quad \text{and} \quad \int_{\Omega} g(x, Y_n) Y_n = o_n(1).$$

Applying a slightly modified version of the Brézis-Lieb lemma [13], one has

$$R(X_n, Y_n) = R(u_n, v_n) - R(u, v) + o_n(1).$$

It follows that

$$\begin{aligned} P(X_n) - R(X_n, Y_n) &= o_n(1), \\ Q(Y_n) - R(X_n, Y_n) &= o_n(1), \\ I_{0,0}(X_n, Y_n) &= c - I_{\lambda, \mu}(u, v) + o_n(1). \end{aligned}$$

Notice that the Nehari manifold associated to $I_{0,0}$ is given by

$$\mathcal{N}_{0,0} = \{(s_0(u, v)u, t_0(u, v)v); (u, v) \in W_{\Gamma_1}^{1,p}(\Omega) \times W_{\Gamma_2}^{1,q}(\Omega), u \not\equiv 0, v \not\equiv 0\},$$

where

$$s_0(u, v) = \left[\frac{P(u)Q(v)^{\frac{r(\beta+1)}{q(\alpha+1)}}}{R(u, v)^{\frac{r}{\alpha+1}}} \right]^{\frac{1}{r-p}}, \quad t_0(u, v) = t(s_0(u, v)),$$

and

$$r = \frac{(\alpha+1)q}{q - (\beta+1)} > p, \quad t(s) = \left[\frac{R(u, v)}{Q(v)} \right]^{\frac{r}{q(\alpha+1)}} s^{\frac{r}{q}}.$$

Let ℓ be the common limit of $P(X_n)$, $Q(Y_n)$ and $R(X_n, Y_n)$. We claim that $\ell = 0$. By contradiction, suppose that $\ell \neq 0$, then on one hand we get

$$\begin{aligned} I_{0,0}(s_0(X_n, Y_n)X_n, t_0(X_n, Y_n)Y_n) &= (\alpha + 1) \left(\frac{1}{p} - \frac{1}{r} \right) K(X_n, Y_n) (2.34) \\ &\geq \inf_{w \in \mathcal{N}_{0,0}} I_{0,0}(w), \end{aligned}$$

where

$$K(X_n, Y_n) = \left[\frac{P(X_n)^{(\alpha+1)} Q(Y_n)^{(\beta+1) \frac{p}{q}}}{R(X_n, Y_n)^p} \right]^{\frac{r}{(\alpha+1)(r-p)}}.$$

A direct computation shows that

$$\lim_{n \rightarrow +\infty} K(X_n, Y_n) = \ell,$$

therefore

$$\lim_{n \rightarrow +\infty} I_{0,0}(s_0(X_n, Y_n)X_n, t_0(X_n, Y_n)Y_n) = \ell(\alpha + 1) \left(\frac{1}{p} - \frac{1}{r} \right).$$

On the other hand,

$$\begin{aligned} \lim_{n \rightarrow +\infty} I_{0,0}(X_n, Y_n) &= \ell \left(\frac{\alpha+1}{p} + \frac{\beta+1}{q} - 1 \right) \\ &= \ell(\alpha + 1) \left(\frac{1}{p} - \frac{1}{r} \right). \end{aligned}$$

Hence, we obtain

$$\ell(\alpha + 1) \left(\frac{1}{p} - \frac{1}{r} \right) = c - I_{\lambda,\mu}(u, v),$$

and consequently

$$\begin{aligned} c &\geq \inf_{w \in \mathcal{N}_{0,0}} I_{0,0}(w) + I_{\lambda,\mu}(u, v) \\ &\geq \inf_{w \in \mathcal{N}_{0,0}} I_{0,0}(w) + \inf_{w \in \mathcal{N}_{\lambda,\mu} \cup \{(0,0)\}} I_{\lambda,\mu}(w). \end{aligned}$$

This leads to a contradiction with (2.29), then $\ell = 0$, which achieves the proof. \square

Remark 2.2 1) In the scalar case, we obtain the analogous of Theorem 2.2, the proof follows easily with the same arguments. We note here that if we consider the special case (1.1), direct computations show that

$$\inf_{w \in \mathcal{N}_0} I_0(w) = \frac{1}{N} S^{\frac{N}{2}} \text{ and } \inf_{w \in \mathcal{N}_\lambda \cup \{0\}} I_\lambda(w) = 0,$$

which recovers the famous Brézis-Nirenberg condition (1.2).

2) It is clear that our condition (2.7) or (2.29) can be extended to a large class of quasilinear or semilinear differential operators: Leray-Lions type operators, fourth-order operators.

3) Using the Hölder inequality in the denominator $R(u, v)$, we get

$$\inf_{(u,v) \in \mathcal{N}_{0,0}} I_{0,0}(u, v) \geq (\alpha + 1) \left(\frac{1}{p} - \frac{1}{r} \right) \left[S_p S_q^{\frac{p(\beta+1)}{q(\alpha+1)}} \right]^{\frac{r}{r-p}}, \quad (2.35)$$

where S_p (resp. S_q) denotes the best Sobolev constant in the embedding $W_{\Gamma_1}^{1,p}(\Omega) \subset L^{p^*}(\Omega)$ (resp. $W_{\Gamma_2}^{1,q}(\Omega) \subset L^{q^*}(\Omega)$).

We end this note by the following interesting relation arising in the special case $p = q$ and $\Gamma_1 = \Gamma_2$.

Proposition 2.1 Assume that $p = q > 1$. Then,

$$\inf_{(u,v) \in \mathcal{N}_{0,0}} I_{0,0}(u, v) = \frac{p}{N-p} S_p^{\frac{N}{p}}.$$

Proof. In the special case $p = q$, direct computations give

$$p^* = \alpha + \beta + 2 \quad \text{and} \quad (\alpha + 1) \left(\frac{1}{p} - \frac{1}{r} \right) = \frac{p}{N-p}.$$

Then, using (2.35), we conclude that

$$\inf_{(u,v) \in \mathcal{N}_{0,0}} I_{0,0}(u, v) \geq \frac{p}{N-p} S_p^{\frac{N}{p}}.$$

On the other hand, let $(u_n) \subset W_{\Gamma_1}^{1,p}(\Omega)$ be a minimizing sequence of S_p . Then using the identity (2.34), we get

$$\begin{aligned} \inf_{w \in \mathcal{N}_{0,0}} I_{0,0}(w) \leq I_{0,0}(s_0(u_n, u_n)u_n, t_0(u_n, u_n)u_n) &= \frac{p}{N-p} \left[\frac{\|\nabla u_n\|_p^p}{\|u_n\|_{p^*}^p} \right]^{\frac{rp^*}{(\alpha+1)(r-p)}} \\ &= \frac{p}{N-p} \left[\frac{\|\nabla u_n\|_p^p}{\|u_n\|_{p^*}^p} \right]^{\frac{N}{p}}. \end{aligned}$$

It is clear that the last quantity goes to $\frac{p}{N-p} S_p^{\frac{N}{p}}$ as $n \rightarrow \infty$, which achieves the proof. \square

Remark 2.3 For the sharpness of the critical level (2.29), we define the sequence $\psi_n(x) := \varphi(x)\Phi_{1/n}(x)$ as in (2.15). We consider then a Palais-Smale sequence (u_n, v_n) for $J_{\lambda, \mu}$ of level $\inf_{w \in \mathcal{N}_{\lambda, \mu} \cup \{(0,0)\}} I_{\lambda, \mu}(w)$. Following the same arguments developed in the scalar case and using Proposition 2.1, we prove that the sequence $(u_n + \psi_n, v_n + \psi_n)$ is a Palais-Smale sequence for $J_{\lambda, \mu}$ of level $c^*(\lambda, \mu)$ and which can not be relatively compact in W . This implies the sharpness of the critical level formula (2.29).

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Khalid Adriouch
 Laboratoire de Math. & Applications,
 Université de la Rochelle,
 17042 La Rochelle, France.
 e-mail: kadriouc@univ-lr.fr

Abdallah El Hamdi
 Laboratoire de Math. & Applications,
 Université de la Rochelle,
 17042 La Rochelle, France.
 e-mail: aelhamid@univ-lr.fr